ABOUT THE UNIVALENCE OF THE BESSEL FUNCTIONS

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Abstract. The authors of [1] and [3] deduced univalence criteria concerning Bessel functions. In [3] the author used the theory developed in [2] to obtain the desired result. In this paper we will extend a few results obtained in [3] employing elementary methods.

1. Introduction

Let

\[ U(z_0, r) = \{ z \in \mathbb{C} : |z - z_0| < r \} \]

be the disc with center \( z_0 \) and of the radius \( r \), the particular case \( U(0, 1) \) will be denoted by \( U \). The Bessel function of the first kind is defined by

\[ J_\nu(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(n + \nu + 1)} \left( \frac{z}{2} \right)^{2n+\nu}. \]

The series, which defines \( J_\nu \) is everywhere convergent and the function defined by the series is generally not univalent in any disc \( U(0, r) \). We will study the univalence of the following normalized form:

\[ f_\nu(z) = 2^\nu \Gamma(1 + \nu) z^{-\frac{\nu}{2}} J_\nu(z^{\frac{1}{2}}), \quad g_\nu(z) = zf_\nu(z). \]

2. Preliminaries

In order to prove our main result we need the following lemmas.
Lemma 1 ([3], equality (6)). The function $f_{\nu}$ satisfies the equality:

$$f_{\nu}'(z) = -\frac{1}{2} f_{\nu+1}(z).$$

Lemma 2. Let $R$ be the function defined by the equality

$$R(\theta) = \sum_{n=3}^{\infty} \frac{(-1)^n (\nu + 1)^n \cos n\theta}{n!(\nu + 1)\ldots(\nu + n)}, \theta \in \mathbb{R}, \nu \in (-1, \infty).$$

The following inequality holds

$$|R(\theta)| \leq \frac{(\nu + 1)^2}{4(\nu + 2)(\nu + 3)}, \theta \in \mathbb{R}.$$

Proof. Since

$$R(\theta) = \frac{\nu + 1}{\nu + 2} \sum_{n=3}^{\infty} \frac{(-1)^n (\nu + 1)^{n-2} \cos n\theta}{n!(\nu + 3)\ldots(\nu + n)}$$

it follows that

$$|R(\theta)| \leq \frac{\nu + 1}{\nu + 2} \sum_{n=3}^{\infty} \left| \frac{(-1)^n (\nu + 1)^{n-2} \cos n\theta}{n!(\nu + 3)\ldots(\nu + n)} \right| \leq \frac{(\nu + 1)^2}{4(\nu + 2)(\nu + 3)}.$$

Lemma 3. If $z \in U$ then

$$|g_{\nu}'(z) - g_{\nu}(z)| \leq \frac{2 + \nu}{(1 + \nu)(4\nu + 7)}, \quad (2)$$

$$|f_{\nu}(z)| = \left| \frac{g_{\nu}(z)}{z} \right| \geq \frac{4\nu^2 + 10\nu + 5}{(1 + \nu)(4\nu + 7)}, \quad (3)$$

$$|f_{\nu}'(z)| \leq \frac{\nu + 2}{(\nu + 1)(4\nu + 7)}. \quad (4)$$

Proof. If $z \in U$ then the triangle inequality implies that:

$$|g_{\nu}'(z) - g_{\nu}(z)| = \left| \sum_{n=1}^{\infty} \frac{(-1)^n n}{4^n n!(\nu + 1)\ldots(\nu + n)} z^n \right| \leq \sum_{n=1}^{\infty} \frac{n}{4^n n!(\nu + 1)\ldots(\nu + n)}.$$
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Since
\[ \sum_{n=1}^{\infty} \frac{n}{4^n n!(\nu + 1)\cdots(\nu + n)} \leq \frac{1}{4(\nu + 1)} \sum_{n=0}^{\infty} \left( \frac{1}{4(\nu + 2)} \right)^n = \frac{2 + \nu}{(1 + \nu)(4\nu + 7)} \]
we obtain (2).

By using again the triangle inequality, we deduce that
\[ \left| \frac{g_{\nu}(z)}{z} \right| \geq 1 - \sum_{n=1}^{\infty} \frac{(-1)^n}{4^n n!(\nu + 1)\cdots(\nu + n)} z^n \geq 1 - \sum_{n=1}^{\infty} \frac{1}{4^n n!(\nu + 1)\cdots(\nu + n)} \]
and so the inequality
\[ 1 - \sum_{n=1}^{\infty} \frac{1}{4^n n!(\nu + 1)\cdots(\nu + n)} \geq 1 - \sum_{n=1}^{\infty} \frac{1}{4(\nu + 1)} \sum_{n=1}^{\infty} \frac{1}{4(n + 2)^{n-1}} = \frac{4\nu^2 + 10\nu + 5}{(1 + \nu)(4\nu + 7)} \]
leads to (3). Using similar ideas we obtain the following inequality chain
\[ |f'_{\nu}(z)| \leq \sum_{n=1}^{\infty} \frac{(-1)^n z^n}{4^n (n-1)!(1 + \nu)\cdots(n + \nu)} \leq \sum_{n=1}^{\infty} \frac{1}{4^n (n-1)!(1 + \nu)\cdots(n + \nu)} \leq \frac{1}{4(1 + \nu)} \sum_{n=0}^{\infty} \left( \frac{1}{4(2 + \nu)} \right)^n = \frac{\nu + 2}{(\nu + 1)(4\nu + 7)}. \]
This means that (4) also holds. □

3. The main result

Theorem 4. If \( \nu > -1 \) then

\[ \Re f_{\nu}(z) > 0, \text{ for all } z \in U(0, 4(1 + \nu)). \]

Proof. The minimum principle for harmonic functions implies that

\[ \Re f_{\nu}(z) \geq \inf_{\theta \in \mathbb{R}} \Re f_{\nu}(r_{\nu} e^{i\theta}), \text{ for all } z \in U(0, 4(1 + \nu)) \]

where \( r_{\nu} = 4(1 + \nu) \). According to the definition of \( f_{\nu} \), we have

\[ f_{\nu}(z) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n z^n}{4^n n!(\nu + 1)\cdots(\nu + n)} \]

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and

\[
\text{Re} f_\nu(r_\nu e^{i\theta}) = 1 + \text{Re} \sum_{n=1}^{\infty} \frac{(-1)^n (\nu + 1)^n e^{in\theta}}{n!(\nu + 1)...(\nu + n)}
\]

\[
= 1 - \cos \theta + \frac{\nu + 1}{2(\nu + 2)} \cos 2\theta + \sum_{n=3}^{\infty} \frac{(-1)^n (\nu + 1)^n \cos n\theta}{n!(\nu + 1)...(\nu + n)}.
\]

If we let

\[
P(\theta) = 1 - \cos \theta + \frac{\nu + 1}{2(\nu + 2)} \cos 2\theta \quad \text{and} \quad R(\theta) = \sum_{n=3}^{\infty} \frac{(-1)^n (\nu + 1)^n \cos n\theta}{n!(\nu + 1)...(\nu + n)}
\]

then

\[
\text{Re} f_\nu(r_\nu e^{i\theta}) = P(\theta) + R(\theta). \quad (5)
\]

A study of the behaviour of the function

\[
P : \mathbb{R} \to \mathbb{R}, \quad P(\theta) = 1 - \cos \theta + \frac{\nu + 1}{2(\nu + 2)} \cos 2\theta
\]

leads to the inequalities

\[
P(\theta) \geq \frac{\nu + 1}{2(\nu + 2)}, \quad \theta \in \mathbb{R}, \ \nu \in (-1, 0) \quad \text{and}
\]

\[
P(\theta) \geq \frac{\nu^2 + 4\nu + 2}{4(\nu + 1)(\nu + 3)}, \quad \theta \in \mathbb{R}, \ \nu \in (0, \infty). \quad (6)
\]

From (5), Lemma 1 and (6) it follows that

\[
\text{Re} f_\nu(r_\nu e^{i\theta}) \geq \min_{\theta \in \mathbb{R}} P(\theta) - \max_{\theta \in \mathbb{R}} R(\theta) \geq 0.
\]

Now Lemma 1 and Theorem 1 imply the following result:

**Theorem 5.** If \( \nu > -2 \) then \( \text{Re} f_\nu'(z) < 0 \) for \( z \in U(0, 4(\nu + 2)) \) and hence \( f_\nu \) is univalent in \( U(0, 4(\nu + 2)) \).

**Remark 6.** Theorem 1 and Theorem 2 improves Lemma 1 and Theorem 1 from [3].

**Theorem 7.** If \( \nu > \frac{-17 + \sqrt{33}}{8} \) then the function \( f_\nu \) is convex in \( U \).

**Proof.** We introduce the notation \( p_1(z) = 1 + \frac{zf_\nu''(z)}{f_\nu'(z)} \). The function \( f_\nu \) is convex if and only if

\[
\text{Re} p_1(z) > 0, \quad z \in U. \quad (7)
\]
It is simple to prove that if
\[ |p_1(z) - 1| < 1, \ z \in U \] (8)
then results (7).

Lemma 1 leads to the equality
\[ |p_1(z) - 1| = \left| \frac{zf'_\nu(z)}{f_{\nu+1}(z)} \right|. \]

In (3) and (4) replacing \( \nu \) by \( \nu + 1 \), we deduce that if \( z \in U \) then
\[ \left| \frac{zf'_\nu(z)}{f_{\nu+1}(z)} \right| \leq \frac{\nu + 3}{4\nu^2 + 18\nu + 19}. \]

Now to prove (7) it is enough to show that \( \frac{\nu + 3}{4\nu^2 + 18\nu + 19} < 1 \), but this is immediately using the condition \( \nu > -\frac{17 + \sqrt{33}}{8} \). \( \square \)

**Theorem 8.** If \( \nu > \frac{\sqrt{3}}{2} - 1 \) then the function \( g_\nu \) defined by (1) is starlike of order \( \frac{1}{2} \) in \( U \).

**Proof.** Let \( p \) be the function defined by the equality \( p_2(z) = \frac{2g_\nu'(z)}{g_\nu(z)} - 1 \).

Since \( \frac{g_\nu(z)}{z} \neq 0 \), \( z \in U \) the function \( p_2 \) is analytic in \( U \) and \( p_2(0) = 1 \). The assertion of Theorem 2 is equivalent to
\[ \Re p_2(z) > 0, \ z \in U. \] (9)

It is simple to prove that if
\[ |p_2(z) - 1| < 1, \ z \in U \] (10)
then results (9).

On the other hand inequalities (2) and (3) lead to
\[ |p_2(z) - 1| = 2 \left| \frac{g_\nu'(z)}{g_\nu(z)} - \frac{g_\nu(z)}{z} \right| < \frac{2(2 + \nu)}{4\nu^2 + 10\nu + 5}, \ z \in U. \]

This means that if \( \frac{2(2 + \nu)}{4\nu^2 + 10\nu + 5} < 1 \) then (8) holds, but this inequality is a consequence of the condition \( \nu > \frac{\sqrt{3}}{2} - 1 \). \( \square \)

**Corollary 9.** If \( \nu > \frac{\sqrt{3}}{2} - 1 \) then the function \( h_\nu \) defined by the equality \( h_\nu(z) = \frac{z^{1-\nu}J_\nu(z)}{z} \) is starlike in \( U \).
The proof of this result is based on Theorem 3 and is similar to the proof of Corollary 2 in [3], hence we do not reproduce it here again.

**Remark 10.** Theorem 3, Theorem 4 and Corollary 1 improves the results of Theorem 2, Theorem 3 and Corollary 2 in [3].

**References**


