MULTI-AGENT SYSTEMS: MODELING AND CONTROL

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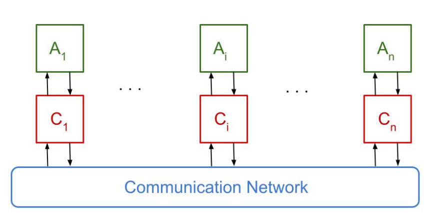
Basic Notions

Definitions

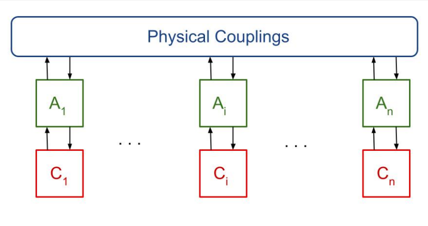
- *Agent (Control Systems view)*: A system with the following proprieties:
    1. Its behavior can be controlled.
    2. It can interact with other agents.

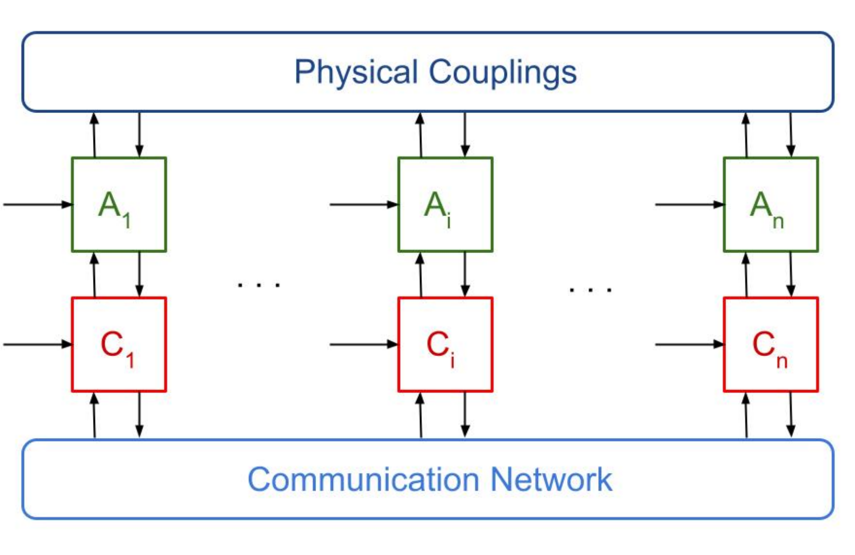
- *Multi-Agent System (Control Systems view)*: A set of agents that interact with one-other (generally to solve a common task).

- Interactions: The state of an agent is influenced by the states of some other agents.
- A_i - Agent
- C_i - Controller



Interactions - Physical Couplings




Physical Couplings
A_1
A_i
A_n
C_1
C_i
C_n
. . .
. . .
Communication Network

Cooperative Control

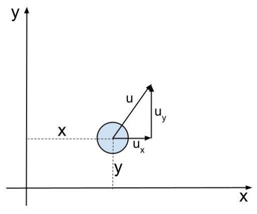
- Cooperation: a process during which a group of agents work together to achieve a *common* goal.
- Cooperation needs interaction.
- Question: Which information is necessary for each agent to achieve the common goal?

Consensus Problem

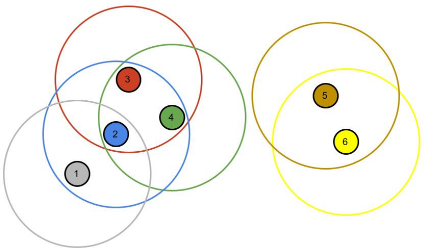
- Consider a robotic agent that moves in a plane and its velocity can be set along the two axis.

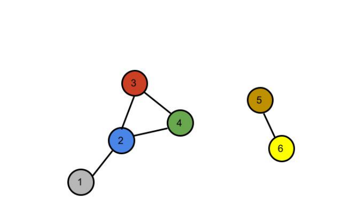
$$\dot{x} = u_x$$
$$\dot{y} = u_y$$

- u_x and u_y: the velocities along the two axis
- Assume that an agent can detect the other agents in its proximity (neighbors) using some kind of obstacle detector with finite range (e.g. LIDAR)

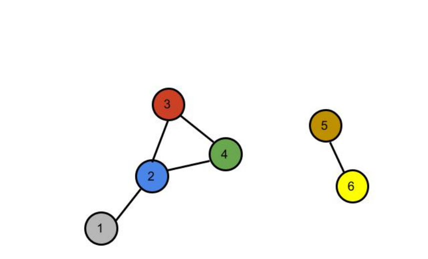


- Consider a swarm of 6 robots equipped with such sensors.
- The interactions in this swarm are limited by the range of the sensors. Based on this, we can define the *proximity graph* of the group.

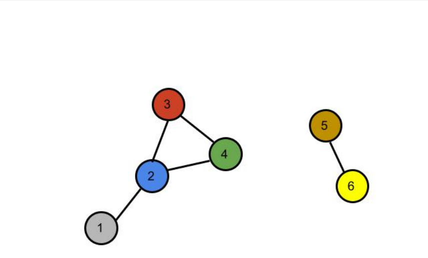




- Undirected graphs: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$
- $\mathcal{V} = \{1, 2, \dots n\}$ - set of vertices.
- $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ - set of edges.
- If the vertices i and j are connected, they are called to be *adjacent*.



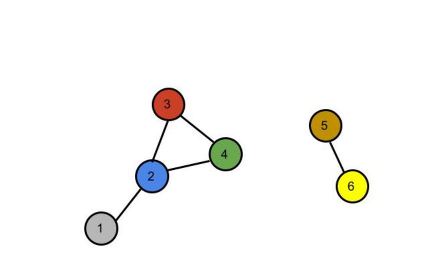
- If the vertices i and j are connected, they are called to be *adjacent*.
- *Neighbor set* of vertex i ($\mathcal{N}_i$): The set of vertices that are adjacent to i
- *Path:* A sequence of adjacent vertices.
- *Strongly Connected graph*: There is a path from any vertex to any other vertex.
- If a graph is not strongly connected it can be decomposed into strongly connected components (subgraphs).



Matrices Associated to Graphs

- Degree matrix: $D = (d_{ij}) = \mathrm{diag}\,(\dim \mathcal{N}_i)$
- Adjacency matrix: $A = (a_{ij}) = \begin{cases} 1, & \text{if } i \text{ and } j \text{ adjacent,} \\ 0, & \text{otherwise.} \end{cases}$

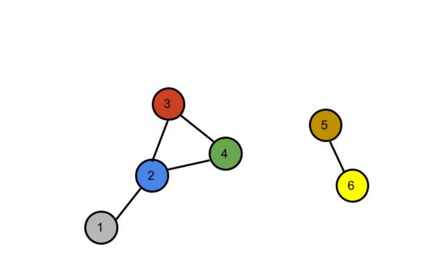
$$D = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$



Laplacian Matrix of a Graph

- Laplacian matrix: $L = D - A$

$$L = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 3 & -1 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & -1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$



Eigenvalues of Laplacian Matrix

- The row sum and column sum are zero, i.e. $L\mathbf{1} = \mathbf{0}$, $\mathbf{1}^T L = \mathbf{0}^T$.
- L has at least one zero eigenvalue and the corresponding eigenvector is $\mathbf{1}$.
- $L = L^T$, i.e. all the eigenvalues are real.
- All the eigenvalues are real, by Greshgorin's theorem, and $\lambda_1 = 0 \leq \lambda_2 \leq \ldots \leq \lambda_n$.
- *The Laplacian of an undirected graph has as many 0 eigenvalues as many strongly connected component the graph has.*
- If the graph is strongly connected, $\mathrm{rank} L = n - 1$.

$$L = \begin{bmatrix} L_1 & O \\ O & L_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 3 & -1 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & -1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

Consensus Problem

- Let a MAS consisting of n agents as: $\dot{x}_i = u_i$, $x_i(0) = x_{i0}$, $i = 1, \ldots, n$.
- The MAS is said to reach consensus if $\forall x_{i0}$, $i = 1, \ldots, n$.

$$\lim_{t\to\infty} x_1(t) = \lim_{t\to\infty} x_2(t) = \ldots = \lim_{t\to\infty} x_n(t) = \bar{x},\ \bar{x} \in \mathbb{R}$$

Example: Rendezvous Problem

- Agent 1 with control: $\dot{x}_1 = u_1$, $x_1(0) = x_{10}$, $u_1 = x_2 - x_1$
- Agent 2 with control: $\dot{x}_2 = u_2$, $x_2(0) = x_{20}$, $u_1 = x_1 - x_2$
- The controlled multi-robot system:

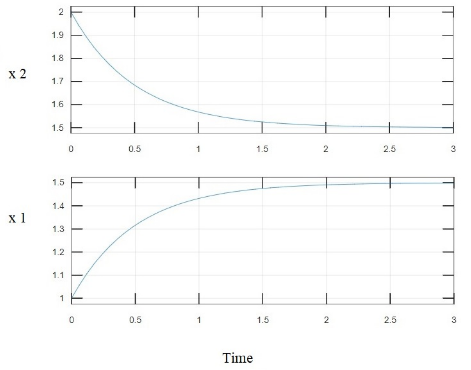
$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = -\underbrace{\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}}_{L} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

- Eigenvalues are $\lambda_1 = 0$, $\lambda_2 = -2$.

Example: Rendezvous Problem

Simulations:

- $x_{10} = 1, \ u_1 = x_2 - x_1$
- $x_{20} = 2, \ u_1 = x_1 - x_2$

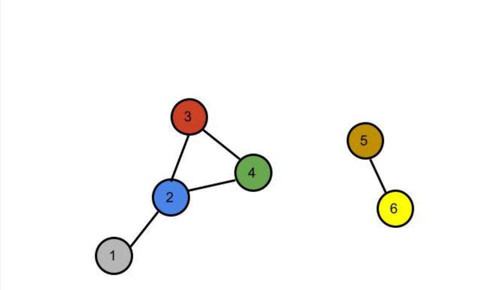


Consensus Protocol

- Let the control (consensus protocol) for each agent $u_i = \sum_{j \in \mathcal{N}_i} (x_j - x_i)$
- The control directs the trajectory of the agent toward the centroid of the neighboring agents.
- The global model of the MAS with consensus protocol:

$$\dot{\mathbf{x}} = -L\mathbf{x},\ \mathbf{x}(0) = \mathbf{x}_0$$
$$\mathbf{x} = (x_1 \ \ldots \ x_n)^T$$

Example: Consensus Protocol

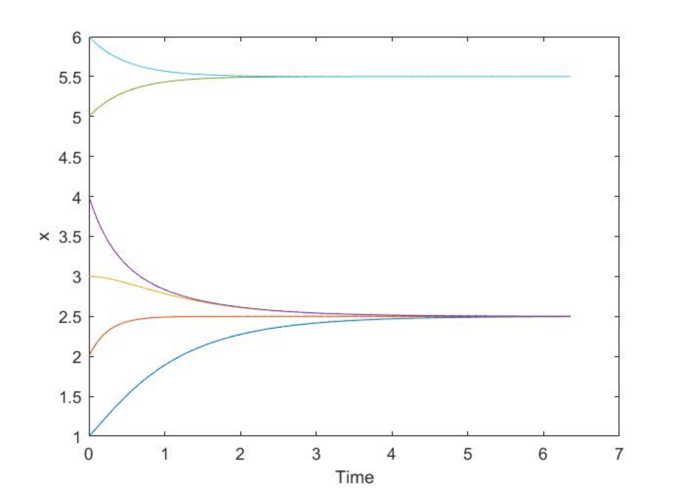


- Control and global MAS model:

$$
\begin{aligned}
u_1 &= x_2 - x_1 \\
u_2 &= x_1 + x_3 + x_4 - 3x_2 \\
u_3 &= x_2 + x_4 - 2x_2 \\
u_4 &= x_3 + x_2 - 2x_4 \\
u_5 &= x_6 - x_5 \\
u_6 &= x_5 - x_6
\end{aligned}
\qquad
\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \\ \dot{x}_6 \end{pmatrix}
= - \underbrace{\begin{bmatrix}
1 & -1 & 0 & 0 & 0 & 0 \\
-1 & 3 & -1 & -1 & 0 & 0 \\
0 & -1 & 2 & -1 & 0 & 0 \\
0 & -1 & -1 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 \\
0 & 0 & 0 & 0 & -1 & 1
\end{bmatrix}}_{L}
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix}
$$

Example: Consensus Protocol

- $eig(L) = [0\ 1\ 3\ 4\ 0\ 2]$



- Question: Does the consensus protocol solves the consensus problem? If yes, under which condition?
- Analyze the system

$$\dot{\mathbf{x}} = -L\mathbf{x}, \ \mathbf{x}(0) = \mathbf{x}_0$$

- The general solution of it, assuming different non-zero eigenvalues:

$$\mathbf{x}(t) = c_1 e^{-\lambda_1 t}\mathbf{v}_1 + c_2 e^{-\lambda_2 t}\mathbf{v}_2 + \ldots + c_n e^{-\lambda_n t}\mathbf{v}_n$$

- Here c_1 are constants and $\mathbf{v}_i$ is the eigenvector corresponding to λ_i:

$$L\mathbf{v}_i = \lambda_i \mathbf{v}_i$$

Analysis of Consensus Protocol

- Let the bock diagonal hypermatrix with two Laplace matrices in the diagonal:

$$L = \begin{bmatrix} L_1 & O \\ O & L_2 \end{bmatrix}. \quad \text{Example}: L = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 3 & -1 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & -1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

- The eigenvalues of L consist of eigenvalues of L_1 and eigenvalues of L_2.
- L_1 has one zero eigenvalue ($\lambda_{11} = 0$) and the corresponding eigenvector of L is

$$\mathbf{v}_{11} = [\mathbf{1}^T\ \mathbf{0}^T]^T \quad \text{Example}: \mathbf{v}_{11} = [1\ 1\ 1\ 1\ 0\ 0]^T$$

- L_2 has one zero eigenvalue ($\lambda_{21} = 0$) and the corresponding eigenvector of L is

$$\mathbf{v}_{21} = [\mathbf{0}^T\ \mathbf{1}^T]^T \quad \text{Example}: \mathbf{v}_{21} = [0\ 0\ 0\ 0\ 1\ 1]^T$$

- The general solution $\dot{\mathbf{x}} = -L\mathbf{x}$, where $L = \mathrm{diag}(L_1 \;\, L_2)$:

$$\mathbf{x}(t) = c_{11}e^{-\lambda_{11}t}\mathbf{v}_{11} + \ldots + c_{1n1}e^{-\lambda_{1n1}t}\mathbf{v}_{1n1} + c_{21}e^{-\lambda_{21}t}\mathbf{v}_{21} + \ldots + c_{2n1}e^{-\lambda_{2n1}t}\mathbf{v}_{2n2}$$

- $\lambda_{11} = 0$, $\lambda_{1i} > 0$, $\lambda_{21} = 0$, $\lambda_{2i} > 0$, $i \geq 2$. Hence:

$$\lim_{t\to\infty}\mathbf{x}(t) = c_{11}\begin{pmatrix} \mathbf{1} \\ \mathbf{0} \end{pmatrix} + c_{21}\begin{pmatrix} \mathbf{0} \\ \mathbf{1} \end{pmatrix} = \begin{pmatrix} c_{11}\mathbf{1} \\ c_{21}\mathbf{1} \end{pmatrix}$$

- Recall the global model of MAS

$$\dot{\mathbf{x}} = -L\mathbf{x},\ \mathbf{x}(0) = \mathbf{x}_0$$

- *The scalar*

$$z = \mathbf{1}^T\mathbf{x}$$

is an invariant quantity along the dynamics of MAS, i.e. $z(t_1) = z(t_2),\ \forall t_1, t_2$.
Proof: $\dot{z} = \mathbf{1}^T\dot{\mathbf{x}} = -\mathbf{1}^T L\mathbf{x} = 0$.

- As a consequence:

$$z(0) = \mathbf{1}^T\mathbf{x}(0) = \lim_{t\to\infty}\mathbf{1}^T\mathbf{x}(t) = z_\infty$$

Consensus Value

- Recall the steady state of the general solution of global MAS model:

$$\dot{\mathbf{x}}_1 = -L_1\mathbf{x}_1,\ \mathbf{x}_1(0) = \mathbf{x}_{01}, \qquad \lim_{t\to\infty}\mathbf{x}_1(t) = c_{11}\mathbf{1}$$

$$\dot{\mathbf{x}}_2 = -L_2\mathbf{x}_2,\ \mathbf{x}_2(0) = \mathbf{x}_{02}, \qquad \lim_{t\to\infty}\mathbf{x}_2(t) = c_{21}\mathbf{1}$$

- Due to the invariant proprieties:

$$\mathbf{1}^T\mathbf{x}_1(0) = \lim_{t\to\infty}\mathbf{1}^T\mathbf{x}_1(t) = c_{11}n_1 \quad \Rightarrow \quad c_{11} = \sum_{i=1}^{n_1} x_{1i}(0)/n_1$$

$$\mathbf{1}^T\mathbf{x}_2(0) = \lim_{t\to\infty}\mathbf{1}^T\mathbf{x}_2(t) = c_{21}n_2 \quad \Rightarrow \quad c_{21} = \sum_{i=1}^{n_2} x_{2i}(0)/n_2$$

- The consensus value:

$$\lim_{t\to\infty}\mathbf{x}_1(t) = \frac{\sum_{i=1}^{n_2} x_{1i}(0)}{n_1}\mathbf{1}$$

$$\lim_{t\to\infty}\mathbf{x}_2(t) = \frac{\sum_{i=1}^{n_2} x_{2i}(0)}{n_2}\mathbf{1}$$

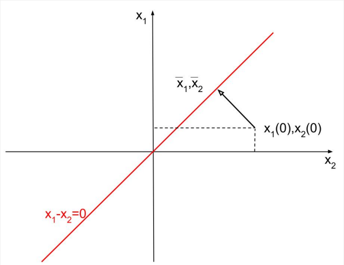
Example: Rendezvous Problem

- The global MAS model:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = -\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

- The space of the steady states:

$$x_1 - x_2 = 0$$

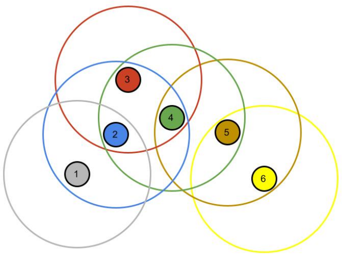


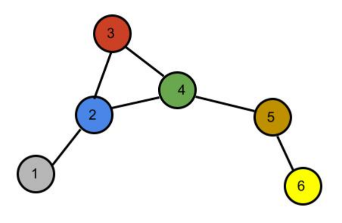
Consensus Problem

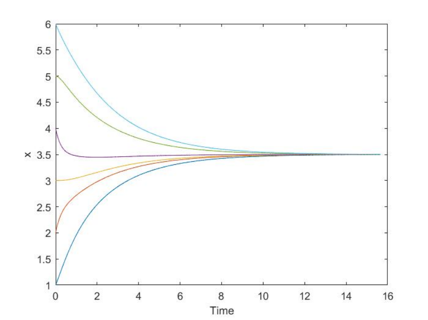
- *The consensus protocol solves the consensus problem of a MAS if the underlying graph of the MAS is strongly connected. The consensus value is:*

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i(0)}{n}$$

Consensus Example







Formation Control

Weak Formation Control Problem

- Let a MAS consisting of n agents as: $\dot{x}_i = u_i$, $x_i(0) = x_{i0}$, $i = 1, \ldots, n$.
- Develop the control (u_i) such that $lim_{t\to\infty}|x_i(t) - x_j(t)| = \delta_{ij}\ \forall x_{i0}$, $i = 1, \ldots, n$, where $\delta_{ij} \geq 0$.
- Remark: The structure of the graph could impose restrictions on δ_{ij}.

Weak Formation Control

- $\delta_{ij} > 0$ is feasible if $\exists p_i,\ p_j$ such that $\delta_{ij} = p_i - p_j\ \forall i, j$.
- Let

$$e_i = x_i - p_i$$

- Weak formation control protocol

$$u_i = \sum_{j \in \mathcal{N}_i} (e_j - e_i)$$

- *The weak formation control protocol solves the consensus problem of a MAS if the underlying graph of the MAS is strongly connected.*
- As $\dot{x}_i = \dot{e}_i$ the transformed model of the global MAS is

$$\dot{\mathbf{e}} = -L\mathbf{e}$$

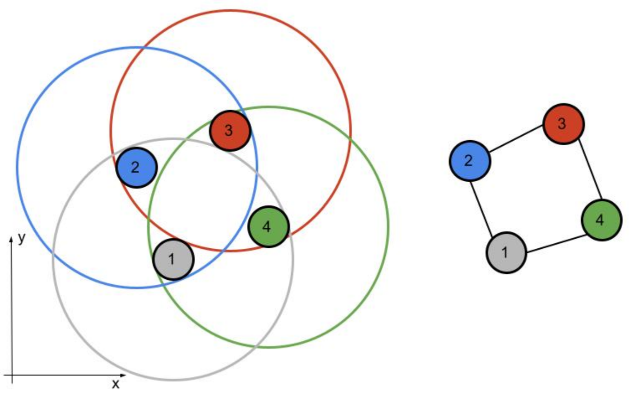
- In the same way as in the case of the consensus problem the weak formation control protocol ensures that

$$\lim_{t\to\infty} e_1(t) = \lim_{t\to\infty} e_2(t) = \ldots = \lim_{t\to\infty} e_n(t) = \bar{e}, \quad \bar{e} = \frac{\sum_{i=1}^{n} e_i(0)}{n}$$

i.e.

$$\lim_{t\to\infty} x_i(t) = p_i + \bar{e},$$
$$lim_{t\to\infty} |x_i(t) - x_j(t)| = |p_i - p_j| = \delta_{ij} \ \ \forall i, j.$$

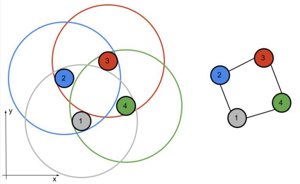
Example: 2DOF Weak Formation Control

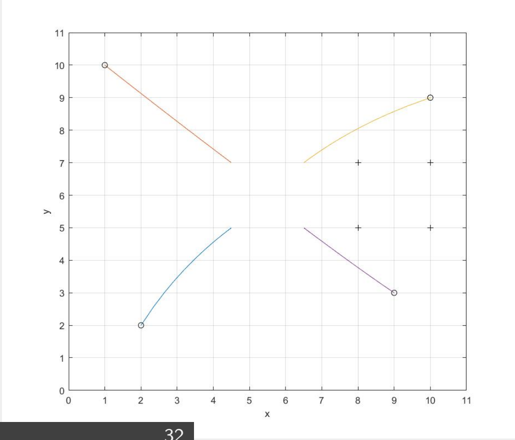


$$\dot{x}_i = u_{ix}$$

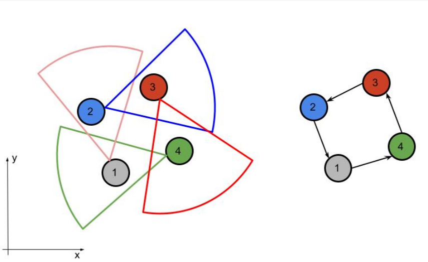
$$\dot{y}_i = u_{iy}$$

Example: 2DOF Weak Formation Control





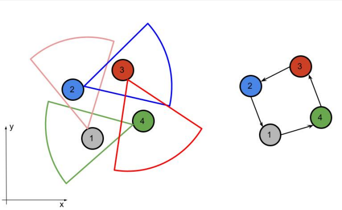
Consensus over Directed Graphs



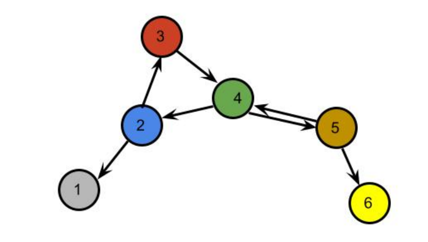
- Assume a number of robotic agents equipped with such obstacle localization sensor which ranges are limited.
- With such sensors the agent *i* may influence the behavior of agent *j* but it could not be true vice-versa.

Directed Graphs

- Directed graphs: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$
- $\mathcal{V} = \{1, 2, \dots n\}$ - set of vertices.
- $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ - set of directed edges.
- Adjacent vertices: j and i are adjacent if there is a directed edge from j to i ($j \rightarrow i$).
- $\mathcal{N}_{Ii}$ *input neighborhood set* of vertex i: Vertex $j \in \mathcal{N}_{Ii}$ if j and i are adjacent ($j \rightarrow i$).
- $\mathcal{N}_{Oi}$ *output neighborhood set* of vertex i: Vertex $j \in \mathcal{N}_{Oi}$ if i and j are adjacent ($i \rightarrow j$).

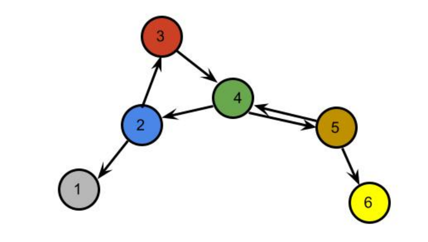


- A *directed path* is a sequence of adjacent vertices $(i_1 \to i_2 \to \ldots \to i_m)$.
- $\mathcal{G}$ is said to posses a *spanning tree* if there exists a vertex r (root) such that there exists a directed path from r to any other vertex j.



Laplacian of a Directed Graph

$$L = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$



Laplacian of a Directed Graph

$$L = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

- The row sum of L is zero.
- It has a zero eigenvalue $(\lambda_1 = 0)$ and the corresponding right eigenvector is $\mathbf{1}$:
$$L\mathbf{1} = \mathbf{0}$$
- The column sum of L is *not* necessarily zero.
- The left eigenvector $(\mathbf{w}_1)$ corresponding $\lambda_1 = 0$ is *not* necessarily equal to $\mathbf{1}$:
$$\mathbf{w}_1^T L = \mathbf{0}^T$$

Consensus over Directed Graphs

- Let a MAS consisting of n agents as: $\dot{x}_i = u_i$, $x_i(0) = x_{i0}$, $i = 1, \ldots, n$.

- *The consensus protocol*

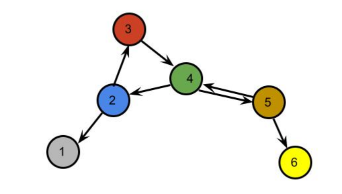
$$u_i = \sum_{j \in \mathcal{N}_{Ii}} (x_j - x_i)$$

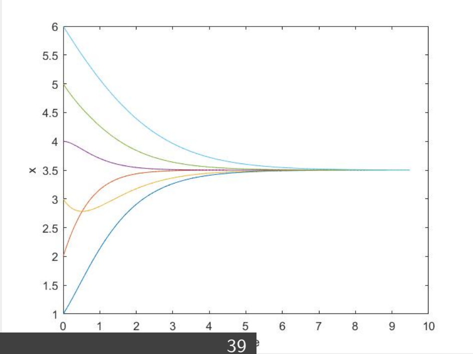
  solves the consensus problem of a MAS if the underlying graph of the MAS possesses a spanning tree.

- The consensus value is:

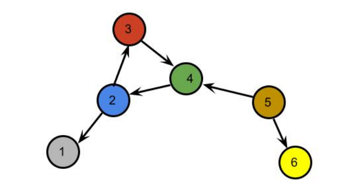
$$\bar{x} = \frac{\mathbf{w}_1^T \mathbf{x}_0}{\mathbf{w}_1^T \mathbf{1}}$$

Example 1: Consensus over Directed Graphs





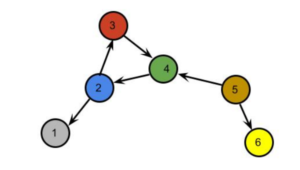
Example 2: Consensus over Directed Graphs

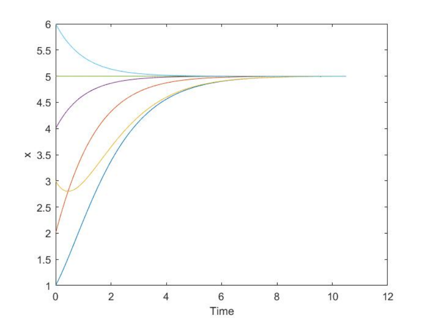


$$L = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

- The left eigenvector corresponding to $\lambda_1 = 0$ is $\mathbf{w}_1 = (0\ 0\ 0\ 0\ \alpha\ 0\ 0)$, $\alpha \in \mathbb{R}$.
- The consensus value is $\bar{x} = \mathbf{w}_1^T \mathbf{x}_0 / \mathbf{w}_1^T \mathbf{1} = \alpha x_{05} / \alpha = x_{05}$.

Example 2:
Consensus over Directed Graphs





Setpoint Control Problem over Graphs

- Control problem: Let a MAS over directed graphs consisting of n agents as: $\dot{x}_i = u_i$, $x_i(0) = x_{i0}$, $i = 1, \ldots, n$.

- Develop the control (u_i) such that $\lim_{t\to\infty} x_i = x_P \; \forall x_{i0}$, $i = 1, \ldots, n$, where $x_P \in \mathbb{R}$ is prescribed.

Setpoint Control over Graphs

- *Leader*: Agent ℓ in a MAS is a leader if $\mathcal{N}_{I\ell} = \emptyset$ and $\mathcal{N}_{O\ell} \neq \emptyset$.
- *Followers*: All the other agents.
- Leader protocol:

$$u_\ell = x_P - x_\ell$$

- Followers protocol is the consensus protocol:

$$u_i = \sum_{j \in \mathcal{N}_{Ii}} (x_j - x_i), \; i \neq \ell$$

- The leader protocol combined the followers protocol solves the setpoint control problem if the MAS has one spanning tree which root is the leader.

Strong Formation Control Problem

- Let a MAS consisting of n agents as: $\dot{x}_i = u_i$, $x_i(0) = x_{i0}$, $i = 1, \ldots, n$.
- Develop the control (u_i) such that $\lim_{t \to \infty} x_i = x_{Pi}$ $\forall x_{i0}$, $i = 1, \ldots, n$, where $x_{Pi} \in \mathbb{R}$ is prescribed.

- Define

$$e_i = x_i - x_{Pi}$$

- Leader protocol:

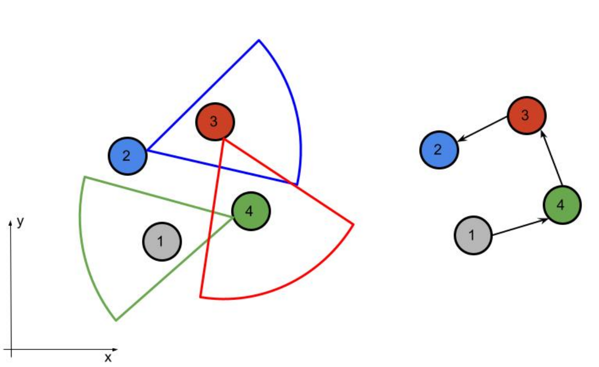
$$u_\ell = -e_\ell$$

- Followers protocol:

$$u_i = \sum_{j \in \mathcal{N}_i} (e_j - e_i), \;\; i \neq \ell$$

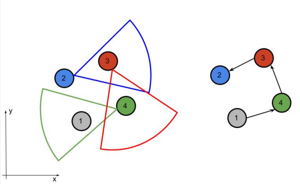
- The leader protocol combined the followers protocol solves the strong formation control problem if the MAS has one spanning tree which root is the leader.

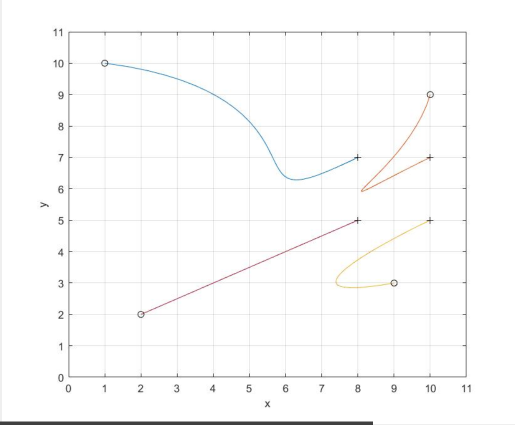
Example: 2DOF Strong Formation Control



$$L = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 1 \end{bmatrix}$$

Example: 2DOF Strong Formation Control





Synchronization of Nonlinear Systems

Model of Nonlinear Dynamic Systems

- Model of nonlinear dynamic systems

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}),\ \ \mathbf{x}(0) = \mathbf{x}_0$$

  such that $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ is smooth.
- $\mathbf{x} \in \mathbb{R}^n$ is the state vector.
- Let $\mathrm{s}(t, \mathbf{x}_0, t_0)$ a *trajectory* of the dynamic system above.

LaSalle's Invariance Principle

- Let nonlinear dynamic system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}),\ \mathbf{x}(0) = \mathbf{x}_0,$$

- $\mathcal{I} \in \mathbb{R}^n$ is an *invariant set* of the system trajectories if $t_0 \geq 0$ and

$$\xi_0 \in \mathcal{I} \implies \mathbf{s}(t, \xi_0, t_0)\ \forall t > t_0$$

- Assign a storage function $S(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}$ to the system that satisfies

$$\begin{aligned} S(\mathbf{x}) &\geq 0\ \forall \mathbf{x}, \\ S(\mathbf{0}) &= 0. \end{aligned}$$

- *LaSalle's theorem*: If $\dot{S}(\mathbf{x}) \leq 0,\ \forall \mathbf{x} \in \mathcal{X} \in \mathbb{R}^n$ then, as $t \to \infty$, the trajectories of the system tend to the largest invariant set inside

$$\mathcal{S} = \left\{ \mathbf{x} \in \mathcal{X} |\ \dot{S}(\mathbf{x}) = 0 \right\}$$

Model of Open Nonlinear Systems

- The model of open dynamic nonlinear systems:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}), \quad \mathbf{x}(0) = \mathbf{x}_0$$
$$\mathbf{y} = \mathbf{h}(\mathbf{x}, \mathbf{u})$$

  such that $\mathbf{f}, \mathbf{h} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ are smooth.
- $\mathbf{u} \in \mathbb{R}^m$ is the vector of inputs
- $\mathbf{y} \in \mathbb{R}^m$ is the vector of outputs

- A system is called *passive*, if there exists a continuously differentiable *storage function* $S : \mathbb{R}^n \to \mathbb{R}$ such that

$$S(\mathbf{x}) \geq 0, \ \forall \mathbf{x},$$
$$S(\mathbf{0}) = 0,$$

$$\dot{S} \leq \mathbf{y}^T \mathbf{u}, \quad \forall \mathbf{u}, \mathbf{x}$$

or equivalently:

$$S(t) \leq S(0) + \int_0^t \mathbf{y}^T(\tau)\mathbf{u}(\tau) d\tau, \quad \forall \mathbf{u}, \mathbf{x}.$$

Passive Input Affine Systems

- Consider the case of nonlinear input-affine systems

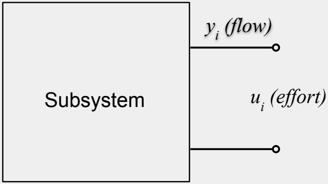
$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}) + G(\mathbf{x})\mathbf{u}, \quad \mathbf{x}(0) = \mathbf{x}_0 \\ \mathbf{y} &= \mathbf{h}(\mathbf{x})\end{aligned}$$

  such that $G : \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^m$ is smooth.
- The input-affine subsystem is passive iff the following conditions hold

$$\begin{aligned}\dot{S} &= \frac{\partial S}{\partial \mathbf{x}}\mathbf{f}(\mathbf{x}) \leq 0, \\ \mathbf{y}^T = \mathbf{h}(\mathbf{x})^T &= \frac{\partial S}{\partial \mathbf{x}} G(\mathbf{x})\end{aligned}$$

Passive Input Affine Systems

- The first condition ($\frac{\partial S}{\partial \mathbf{x}}\mathbf{f}(\mathbf{x}) \leq 0$) prescribes that the "stored energy" of the system is non-increasing if $\mathbf{u} = \mathbf{0}$.
- The second condition ($\mathbf{y}^T = \mathbf{h}(\mathbf{x})^T = \frac{\partial S}{\partial \mathbf{x}} G(\mathbf{x})$) restricts the passive output of the system.
- The inputs and the passive output should be "power-correlated". If the input is an effort (e.g. voltage, force), the output is a flow (e.g. current, velocity).



Example of Passive System

- Let the dynamic model of a mechanical system (x -position, v - velocity, u - external force, m - mass, F_V - damping coefficient)

$$\begin{pmatrix} \dot{x} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 \\ -F_V v/m \end{pmatrix} + \begin{pmatrix} 0 \\ 1/m \end{pmatrix} u$$

- Consider the storage function:

$$S = \frac{mv^2}{2}$$

- Time derivative of the storage function

$$\dot{S} = -F_V v^2 + vu \leq vu$$

- The passive output:

$$y = \frac{\partial S}{\partial \mathbf{x}} G(\mathbf{x}) = (0 \ mv) \begin{pmatrix} 0 \\ 1/m \end{pmatrix} = v$$

- Consider a MAS in which each agent's dynamics is given by

$$\Sigma_i : \begin{cases} \dot{\mathbf{x}}_i = \mathbf{f}_i(\mathbf{x}_i) + G_i(\mathbf{x}_i)\mathbf{u}_i, \quad \mathbf{x}_i(0) = \mathbf{x}_{i0} \\ \mathbf{y}_i = \mathbf{h}_i(\mathbf{x}_i) \end{cases}$$

  such that $G : \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^m$ is smooth.

- Consider that the underlying graph of the network is undirected.

- *Neighborhood set ($\mathcal{N}_i$)*: If $\Sigma_j \in \mathcal{N}_i$ if $\mathbf{u}_i$ may depend on $\mathbf{y}_j$.

- Let a MAS consisting of nonlinear agents $\Sigma_i, \;\; i = 1, \ldots n$.

- The MAS is called *output synchronized* if

$$\lim_{t \to \infty} \|\mathbf{y}_i - \mathbf{y}_j\| = 0, \;\; \forall i, j$$

Let a MAS consisting of nonlinear agents $\Sigma_i,\ i = 1, \ldots n$. If

- Σ_i are passive $\forall i$ (wrt. to storage function $S_i(\mathbf{x}_i)$) and
- the underlying graph is strongly connected,

then the synchronization protocol

$$\mathbf{u}_i = \sum_{j \in \mathcal{N}_i} (\mathbf{y}_j - \mathbf{y}_i)$$

solves the output synchronization problem.

Proof:

- Let the storage function of MAS:

$$S = 2\sum_{i=1}^{n} S_i$$

- The time derivative of the storage function reads as:

$$\dot{S} = 2\sum_{i=1}^{n} \dot{S}_i = 2\sum_{i=1}^{n} \left(\frac{\partial S_i}{\partial \mathbf{x}_i}\mathbf{f}_i(\mathbf{x}_i) + \frac{\partial S_i}{\partial \mathbf{x}_i} G_i(\mathbf{x}_i)\mathbf{u}_i \right)$$

- As the system is passive $\frac{\partial S_i}{\partial \mathbf{x}_i}\mathbf{f}_i(\mathbf{x}_i) \leq 0$ and $\frac{\partial S_i}{\partial \mathbf{x}_i} G_i(\mathbf{x}_i) = \mathbf{h}_i(\mathbf{x}_i)^T = \mathbf{y}_i^T$. Then

$$\dot{S} \leq 2\sum_{i=1}^{n} \mathbf{y}_i^T \mathbf{u}_i$$

Proof:

- The time derivative of the storage function

$$\dot{S} \le 2\sum_{i=1}^{n} \mathbf{y}_i^T \mathbf{u}_i$$

- By applying the synchronization protocol

$$\dot{S} \le 2\sum_{i=1}^{n}\sum_{j\in\mathcal{N}_i} \mathbf{y}_i^T(\mathbf{y}_j - \mathbf{y}_i) = 2\sum_{i=1}^{n}\sum_{j\in\mathcal{N}_i} \mathbf{y}_i^T\mathbf{y}_j - 2\sum_{i=1}^{n}\sum_{j\in\mathcal{N}_i} \mathbf{y}_i^T\mathbf{y}_i$$

- As the graph is strongly connected

$$\sum_{i=1}^{n}\sum_{j\in\mathcal{N}_i} \mathbf{y}_i^T\mathbf{y}_i = \sum_{i=1}^{n}\sum_{j\in\mathcal{N}_i} \mathbf{y}_j^T\mathbf{y}_j$$

- It yields that

$$\dot{S} \le 2\sum_{i=1}^{n}\sum_{j\in\mathcal{N}_i} \mathbf{y}_i^T\mathbf{y}_j - \sum_{i=1}^{n}\sum_{j\in\mathcal{N}_i} \mathbf{y}_i^T\mathbf{y}_i - \sum_{i=1}^{n}\sum_{j\in\mathcal{N}_i} \mathbf{y}_j^T\mathbf{y}_j = -\sum_{i=1}^{n}\sum_{j\in\mathcal{N}_i} (\mathbf{y}_i - \mathbf{y}_j)^T(\mathbf{y}_i - \mathbf{y}_j)$$

Proof:

- The time derivative of the storage function

$$\dot{S} \leq -\sum_{i=1}^{n} \sum_{j \in \mathcal{N}_i} (\mathbf{y}_i - \mathbf{y}_j)^T (\mathbf{y}_i - \mathbf{y}_j)$$

- As $\dot{S} \leq 0$ we can apply the LaSalle theorem.
- The largest invariant set defined by $\dot{S} = 0$ is given by

$$(\mathbf{y}_i - \mathbf{y}_j)^T (\mathbf{y}_i - \mathbf{y}_j) = 0, \ j \in \mathcal{N}_i, \ i = 1 \dots n$$

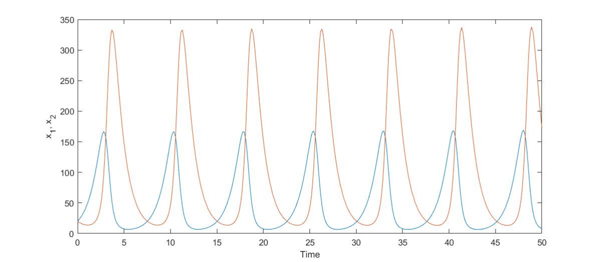
- Hence the synchronization protocol $\mathbf{u}_i = \sum_{j \in \mathcal{N}_i} (\mathbf{y}_j - \mathbf{y}_i)$ solves the synchronization problem of passive nonlinear systems over strongly connected graphs.

Networks Lotka-Volterra Systems

Describes the dynamic behavior of coexisting predator and prey populations:

$$\dot{x}_1 = x_1(l_1 - m_{12}x_2)$$
$$\dot{x}_2 = x_2(-l_2 + m_{21}x_1)$$

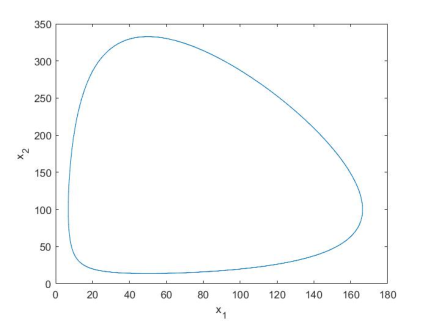
- $x_1 \in \mathbb{R}_{>0}$ - prey population size,
- $x_2 \in \mathbb{R}_{>0}$ - predator population size
- $l_1, l_2 \in \mathbb{R}_{>0}$ - constant prey birth rate and predator death rate
- $m_{12}, m_{21} \in \mathbb{R}_{>0}$ inter-influence parameters of the predator and prey populations



Lotka-Volterra modell

$$\dot{x}_1 = x_1(l_1 - m_{12}x_2)$$
$$\dot{x}_2 = x_2(-l_2 + m_{21}x_1)$$

- Trivial equilibrium point: $\mathbf{x}_0^* = (0\ 0)^T$,
- Non-trivial equilibrium point: $\mathbf{x}^* = (\frac{l_1}{m_{12}}\ \frac{l_3}{m_{21}})^T$



Storage Function for Lotka-Volterra Systems

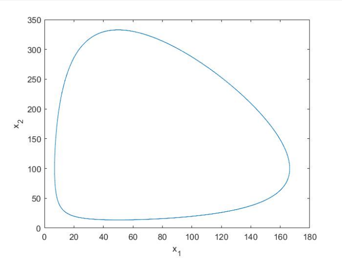
- Storage function:

$$S = c_1\left(x_1 - x_1^* - x_1^*\ln\frac{x_1}{x_1^*}\right) + c_2\left(x_2 - x_2^* - x_2^*\ln\frac{x_2}{x_2^*}\right)$$

where $c_1 = c > 0$ and $c_2 = c\frac{m_{12}}{m_{21}}$

- Time derivative of the storage function

$$\dot{S} = 0$$



Describes the dynamic behavior of coexisting predator and prey population with population in- and outflow:

$$\dot{x}_1 = x_1(l_1 - m_{12}x_2 + u_1)$$
$$\dot{x}_2 = x_2(-l_2 + m_{21}x_1 + u_2)$$

- u_1 - prey population in- or outflow rate,
- u_2 - predator population in- or outflow rate

$$\mathbf{u} = (u_1 \;\; u_2)^T$$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \underbrace{\begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix} \begin{pmatrix} l_1 - m_{12}x_2 \\ -l_2 + m_{21}x_1 \end{pmatrix}}_{\mathbf{f}(\mathbf{x})} + \underbrace{\begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix}}_{G(\mathbf{x})} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

- Storage function:

$$S = c_1\left(x_1 - x_1^* - x_1^* \ln\frac{x_1}{x_1^*}\right) + c_2\left(x_2 - x_2^* - x_2^* \ln\frac{x_2}{x_2^*}\right)$$

- First passivity condition: ($\dot{S} \leq 0$)

$$\dot{S} = 0$$

- Second passivity condition ($\mathbf{y}^T = \mathbf{h}(\mathbf{x})^T = \frac{\partial S}{\partial \mathbf{x}} G(\mathbf{x})$):

$$\mathbf{y}^T = \begin{pmatrix} c_1(x_1 - x_1^*) & c_2(x_2 - x_2^*) \end{pmatrix}$$

- Let a MAS consisting of N agents (habitats) described by open Lotka-Volterra systems:

$$\begin{aligned}\dot{x}_{1i} &= x_{1i}(l_{1i} - m_i x_{1i} + u_{1i}) \\ \dot{x}_{2i} &= x_{2i}(-l_{2i} + m_i x_{2i} + u_{2i})\end{aligned}$$

- As $m_{12i} = m_{21i} = m_i > 0$, the passive output of the agent is

$$\mathbf{y}^T = (x_1 - x_1^* \quad x_2 - x_2^*)$$

- The population flow rate among the agents is driven by the population size difference among them:

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \sum_{j\in\mathcal{N}_i}(x_{1j} - x_{1i}) \\ \sum_{j\in\mathcal{N}_i}(x_{2j} - x_{2i}) \end{pmatrix}$$

- The passive outputs:

$$y_{1i} = x_{1i} - x^*_{1i}$$
$$y_{2i} = x_{2i} - x^*_{2i}$$

- The dynamics of agents in terms of passive outputs:

$$\dot{y}_{1i} = (y_{1i} + x^*_{1i})(l_{1i} - m_i x^*_{1i} - m_i y_{1i} + u_{1i})$$
$$\dot{y}_{2i} = (y_{2i} + x^*_{2i})(-l_{2i} + m_i x^*_{21i} + m_i y_{2i} + u_{2i})$$

- The population flow rate among the agents in terms of passive outputs:

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \sum_{j\in\mathcal{N}_i}(y_{1j} - y_{1i}) + \sum_{j\in\mathcal{N}_i}(x^*_{1j} - x^*_{1i}) \\ \sum_{j\in\mathcal{N}_i}(y_{2j} - y_{2i}) + \sum_{j\in\mathcal{N}_i}(x^*_{2j} - x^*_{2i}) \end{pmatrix}$$

Networks of Lotka-Volterra Systems

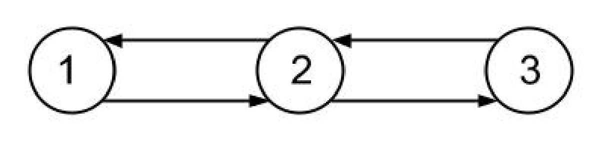
- The dynamics of agents with synchronization protocol:

$$\dot{y}_{1i} = (y_{1i} + x_{1i}^*)(l_{1yi} - m_i y_{1i} + u_{1i}) \quad u_{1i} = \sum_{j \in \mathcal{N}_i} (y_{1j} - y_{1i})$$
$$\dot{y}_{2i} = (y_{2i} + x_{2i}^*)(-l_{2yi} + m_i y_{2i} + u_{2i}) \quad u_{2i} = \sum_{j \in \mathcal{N}_i} (y_{2j} - y_{2i})$$

  where $l_{1yi} = l_{1i} - m_i x_{1i}^* + \sum_{j \in \mathcal{N}_i} (x_{1j}^* - x_{1i}^*)$ and
  $l_{2yi} = l_{2i} - m_i x_{2i}^* + \sum_{j \in \mathcal{N}_i} (x_{2j}^* - x_{2i}^*)$.
- The Lotka-Volterra agents are passive.
- Assume that the underlying graph of the network is strongly connected.
- Hence the synchronization protocol ensures the synchronization of the nonlinear Lotka-Volterra agents, i.e.

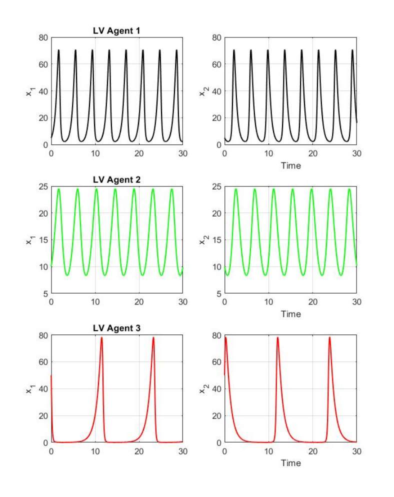
$$\lim_{t\to\infty}(x_{11}(t) - x_{11}^*(t)) = \lim_{t\to\infty}(x_{12}(t) - x_{12}^*(t)) = \ldots = \lim_{t\to\infty}(x_{1n}(t) - x_{1n}^*(t))$$
$$\lim_{t\to\infty}(x_{21}(t) - x_{21}^*(t)) = \lim_{t\to\infty}(x_{22}(t) - x_{22}^*(t)) = \ldots = \lim_{t\to\infty}(x_{2n}(t) - x_{2n}^*(t))$$

Example: Lotka-Volterra Networks

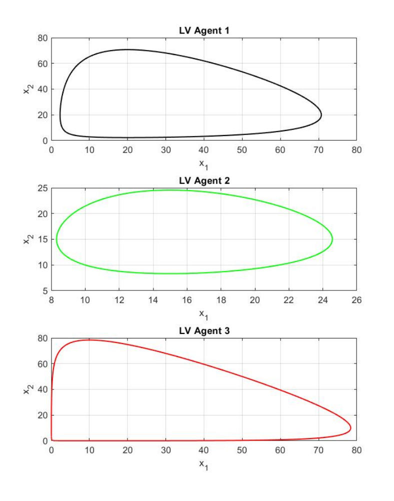


- $x_{11}^* = x_{12}^* = 20$
- $x_{21}^* = x_{22}^* = 15$
- $x_{31}^* = x_{32}^* = 10$

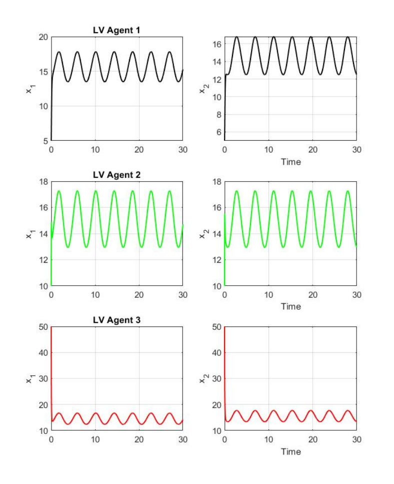
Example: Lotka-Volterra Networks - No Connections



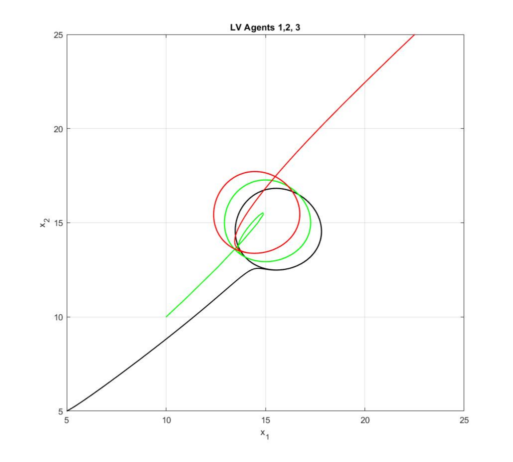
Example: Lotka-Volterra Networks - No Connections



Example: Lotka-Volterra Networks - Connections



Example: Lotka-Volterra Networks - Connections



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