Dominating sets in de Bruijn graphs

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Abstract

In this paper we deal with different type of dominating sets in de Bruijn graphs and we prove a conjecture on perfect dominating sets.

1 Introduction

Let $|\mathcal{A}| = n$. The de Bruijn graph is defined as:

$$B(n,k) = (V(n,k), E(n,k))$$

with $V(n,k) = \mathcal{A}^k$ as the set of vertices, and $E(n,k) = \mathcal{A}^{k+1}$ as the set of directed arcs. There is an arc from $x_1x_2\ldots x_k$ to $y_1y_2\ldots y_k$ if $x_2x_3\ldots x_k = y_1y_2\ldots y_{k-1}$. For $B(2,3)$ see Fig. 1.

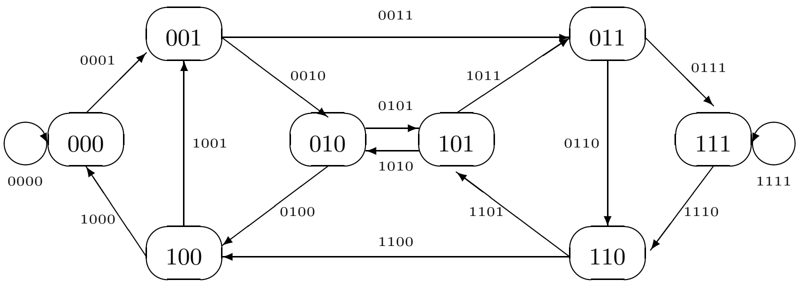

Figure 1: The de Bruijn graph $B(2,3)$.

In a graph $G = (V,E)$ a vertex y is *dominated* by a vertex x (or x *dominates* y) if there exists an arc from x to y or $x = y$. A set of vertices $D \subseteq V$ is a *dominating set* of G if every vertex of G is dominated by at least one vertex of D. The size of a set of least cardinality among all dominating sets for G is called the *domination number* of G and any dominating set of this cardinality is called a *minimum dominating set* for G. When each vertex of G is dominated by exactly one element of D then the set D is called a *perfect dominating set* of G. A vertex x *d-dominate* a vertex y if there is a path from x to y in G of length at most d. A set D of vertices is a *d-dominating set* in G if each vertex of G is d-dominated by at least one vertex of D. This set D is a *perfect d-dominating set (d-PDS)* if each vertex of G is d-dominated by exactly one vertex of D.

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Proposition 1. *In the de Bruijn graph $B(2,k)$ a minimum dominating set has* $\left\lceil \frac{2^k}{3} \right\rceil$ *vertices.*

Proof. Because of the two outgoing arcs from each vertex in $B(2,k)$, a vertex can dominate at most two other vertices. So, a dominating set has at least $\lceil 2^k/3 \rceil$ vertices. We give a dominating set in a de Bruijn graph, proves that this number is enough. Let us consider any vertex in the graph as a binary representation of a natural number and let us refer to it by its numerical value. So, the graph $B(2,k)$ has the following vertices: 0, 1, 2, $\ldots 2^k - 1$. In the case $k = 3$ the dominating set has 3 vertices. The vertex 4 dominates the vertices 0 and 1, the vertex 5 dominates 2 and 3 and finally 7 dominates 6. These result from the following:

4 dominates	0, 1
5 dominates	2, 3
6 dominates	4, 5 – eliminated (4 and 5 were already considered)
7 dominates	6

In the general case we get the following values:

2^{k-1}	dominates	0 and 1
$2^{k-1}+1$	dominates	2 and 3
...		
$2^{k-1}+2^{k-2}-1$	dominates	$2^{k-1}-2$ and $2^{k-1}-1$
2^{k-3} *values eliminated*		
$2^{k-1}+2^{k-2}+2^{k-3}$	dominates	$2^{k-1}+2^{k-2}$ and $2^{k-1}+2^{k-2}+1$
...		
$2^{k-1}+2^{k-2}+2^{k-3}+2^{k-4}-1$	dominates	$2^{k-1}+2^{k-2}+2^{k-3}$ and $2^{k-1}+2^{k-2}+2^{k-3}+1$
2^{k-5} *values eliminated*		
...		
2^k-1	dominates	2^k-2 if k is odd, nothing otherwise.

So, this dominating set has $2^k - (2^{k-1} + 2^{k-3} + 2^{k-5} + \ldots + 2^{k-\lceil k/2 \rceil + 1})$ vertices. It is easy to see (e.g. by induction) that this sum is equal to $\lceil 2^k/3 \rceil$.

A natural generalization of this proposition is the following.

Proposition 2. *In the de Bruijn graph $B(n,k)$ a minimum dominating set has* $\left\lceil \frac{n^k}{n+1} \right\rceil$ *vertices.*

M. Livingston and Q. F. Stout proved in [2] the following result (Theorem 2.12).

Proposition 3. *For any $d \geq 1$ and for k a positive integer of the form $(d+1)m$ or $(d+1)m-1$ or $k < d$, let T_k denote a subset of the vertices of $B(2,k)$ defined as follows.*
(i) $T_1 = T_2 = \ldots = T_k = \{0\}$,
(ii) $T_{(d+1)(m+1)-1} = T_{(d+1)m-1} \cup \{j : 2^{(d+1)m-1} \leq j \leq 2^{(d+1)m} - 1\}$,
(iii) $T_{(d+1)m} = T_{(d+1)m-1} \cup \{2^{(d+1))m} - 1 - s : s \in T_{(d+1)m-1}\}$.
Then the set T_k is a perfect d-dominating set for $B(2,k)$.

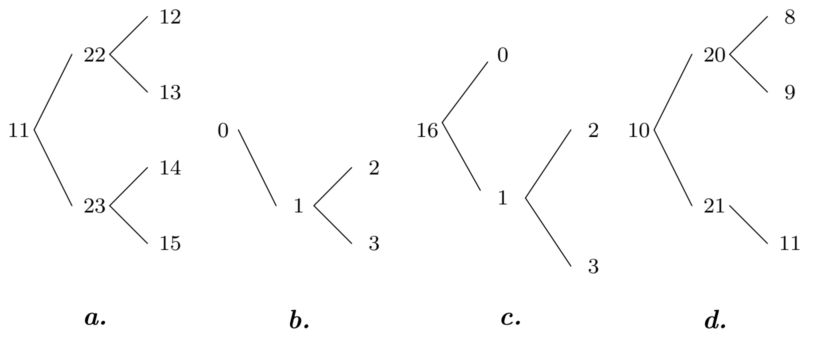

Figure 2: Typical sets in 2-dominations in a de Bruijn graph ($k = 5$)

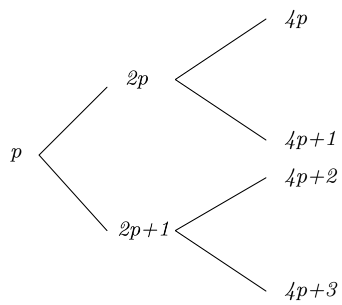

Figure 3: A general set in 2-dominations in a de Bruijn graph. Values are taken mod 2^k.

In [2] the following conjecture was set too: *there is no perfect 2-dominating set for $B(2, k)$, when $k - 1$ is a multiple of 3.* We prove this conjecture in the following proposition.

Proposition 4. *In the de Bruijn graph $B(2, k)$ there is no perfect 2-dominating set if $k - 1$ is a multiple of 3.*

Proof. Let $N_d(G, v)$ denote the set of vertices in the graph $G = (V, E)$ within distance d of vertex v. If D is a perfect d-dominating set for G then $\{N_d(G, v) \mid v \in D\}$ forms a partition of V. Let $n_d(G, v) = |N_d(G, v)|$ then $\sum_{v \in D} n_d(G, v) = |V|$.

Property 1. $n_2(B(2, k), v) = 4, 5, 6$ or 7.

$n_2(B(2, k)), v)$ can differ from 7 only in the following cases (see Fig. 3).
$p = 2p \Rightarrow p = 0 \Rightarrow n_2(B(2, k), 0) = 4$
$p + 2^k = 2p + 1 \Rightarrow p = 2^k - 1 \Rightarrow n_2(B(2, k), 2^k - 1) = 4$
$2p + 2^k = 4p \Rightarrow p = 2^{k-1} \Rightarrow n_2(B(2, k), 2^{k-1}) = 5$
$2p + 2^k = 4p + 2 \Rightarrow p = 2^{k-1} - 1 \Rightarrow n_2(B(2, k), 2^{k-1} - 1) = 5$

If k is odd:
$p + 2^k = 4p + 2 \Rightarrow p = \frac{2^k - 2}{3} \Rightarrow n_2(B(2, k), \frac{2^k - 2}{3}) = 6$
$p + 2^{k+1} = 4p + 1 \Rightarrow p = \frac{2^{k+1} - 1}{3} \Rightarrow n_2(B(2, k), \frac{2^{k+1} - 1}{3}) = 6$
If k is even:
$p + 2^k = 4p + 1 \Rightarrow p = \frac{2^k - 1}{3} \Rightarrow n_2(B(2, k), \frac{2^k - 1}{3}) = 6$
$p + 2^{k+1} = 4p + 2 \Rightarrow p = \frac{2^{k+1} - 2}{3} \Rightarrow n_2(B(2, k), \frac{2^{k+1} - 2}{3}) = 6$

Property 2. In $B(2,k)$ if p d-dominates q, and q is even, then p d-dominates $(q+1)$, too. If p d-dominates q, and q is odd, then p d-dominates $(q-1)$, too.

Property 3. If $n_2(B(2,k),v) = 5$ then v cannot be in a 2-PDS.

This can be proved by direct computations. $n_2(B(2,k),v) = 5$ only for $v = 2^{k-1}$ and $v = 2^{k-1} - 1$.
a) If 2^{k-1} is in a 2-PDS then $2^{k-1} + 1$ cannot be in the 2-PDS, otherwise 2 is 2-dominated by both of them. But if $2^{k-1} + 1$ is not in the 2-PDS then it is 2-dominated together with 2^{k-1}, so 2^{k-1} is dominated by two vertices.
b) If $2^{k-1} - 1$ is in a 2-PDS then $2^{k-1} - 2$ cannot be in the 2-PDS, otherwise $2^k - 4$ is 2-dominated by both of them. But if $2^{k-1} - 2$ is not in a 2-PDS then it is 2-dominated together with $2^{k-1} - 1$, so $2^{k-1} - 1$ is dominated by two vertices.

Property 4. In a 2-PDS there is at most one vertex v for which $n_2(B(2,k),v) = 6$.
For example $\frac{2^k - 1}{3}$ and $\frac{2^{k+1} - 2}{3}$ dominate each other.

Now we suppose that $k = 3l + 1$ and $l > 0$. If we denote the number of different type of vertices in a 2-PDS by A, B and C, we have the equation:

$$4A + 6B + 7C = 2^{3l+1} = 2((2^3)^l - 1) + 2$$

and we know based on the above properties that A is 0, 1 or 2 and B is 0 or 1. If we construct a matrix from the values of the expression $4A + 6B - 2$ we can see that these are not divided by 7. So we got the fact that there is no 2-PDS in $B(2, 3l + 1)$.

M. Livingston and Q. F. Stout in [2] deal with the undirected case, too. We can define the undirected version of the de Bruijn graph $B(n,k)$ if we change arcs to undirected edges. Let us denote by $B^*(n,k)$ the undirected de Bruijn graph. Now there is an edge between $x_1x_2\ldots x_k$ and $y_1y_2\ldots y_k$ if $x_2x_3\ldots x_k = y_1y_2\ldots y_{k-1}$ or $x_1x_2\ldots x_{k-1} = y_2y_3\ldots y_k$. We can give all definitions about *domination* (and *d-domination*) in a very similar way. In [2] we can read the fact that the undirected de Bruijn graph $B^*(2,k)$ has a perfect dominating set (PDS) for k=1 or 2, but has no PDS for k=3, 4 or 5. In the following we prove that $B^*(2,k)$ has PDS only for k=1 or 2.

Property 5. $n_1(B^*(2,k),v) = 3, 4$ or 5.
We consider any vertex in the graph $B^*(2,k)$, if we like it, too, as a binary representation of a natural number.So $N_1(B^*(2,k),p) = \Big\{p, 2p, 2p + 1, \lfloor p/2 \rfloor, 2^{k-1} + \lfloor p/2 \rfloor\Big\}$ where every number are taken mod 2^k. Obviously, 0 and $2^k - 1$ has only "islands" with three vertices. Eg. 0 dominates $0, 1$ and 2^{k-1}.There are two vertices with "islands" of size four: $n_1(B^*(2,k), 2^{k-2} + 2^{k-4} + \ldots + 2) = 4$ and $n_1(B^*(2,k), 2^{k-1} + 2^{k-3} + \ldots + 1) = 4$ if k is an odd number and $n_1(B^*(2,k), 2^{k-2} + 2^{k-4} + \ldots + 1) = 4$ and $n_1(B^*(2,k), 2^{k-1} + 2^{k-3} + \ldots + 2) = 4$ if k is an even number. In other words $B^*(2,k)$ has two loops and two parallel edges between the alternate 0-1 words (see Fig. 1). Of course $n_1(B^*(2,k),v) = 5$ for all other vertices.

Property 6. There is only at most one vertex in a PDS with an "island" of size four.
This is obviously because of the structure of the graph.

Proposition 5. *There is no PDS in $B^*(2,k)$ if $k > 2$.*

Proof. Suppose that we have a PDS D in $B^*(2,k)$. We deal with four cases based on k mod 4. There is only one possibility for sizes of "islands" in all cases according to our properties:

a) $3+3+5t=2^k$ if $k=4l$

b) $3+4+5t=2^k$ if $k=4l+1$

c) $4+5t=2^k$ if $k=4l+2$

d) $3+5t=2^k$ if $k=4l+3$.

In the first case *a*) there is no "island" of size 4. So the two alternating words is not in D, $2^{k-2}+2^{k-4}+\ldots+2^2+1 \notin D$ and $2^{k-1}+2^{k-3}+\ldots+2 \notin D$. We should dominate these words and there is only two possibilities: one of them is $2^{k-3}+2^{k-5}+\ldots+2 \in D$ and $2^{k-1}+2^{k-2}+2^{k-4}+\ldots+2^2+1 \in D$. The other case is very similar. There is a symmetry between the words by exchanging the letters (0 to 1 and 1 to 0), or by read the words from the left to right and back. We use these simplification and deal with only this case. Consider the word $2^{k-3}+2^{k-5}+\ldots+2+1$! We can not put it to D, because $2^{k-3}+2^{k-5}+\ldots+2$ dominates $2^{k-4}+2^{k-6}+\ldots+1$ and $2^{k-3}+2^{k-5}+\ldots+2+1$ dominates it, too. We can not put in D neither $2^{k-4}+2^{k-6}+\ldots+1$ nor $2^{k-1}+2^{k-4}+2^{k-6}+\ldots+1$ because these are dominated by $2^{k-1}+2^{k-3}+\ldots+2$. But the other two neighbours of $2^{k-3}+2^{k-5}+\ldots+2+1$ are $2^{k-2}+2^{k-4}+\ldots+2^2+2$ and $2^{k-2}+2^{k-4}+\ldots+2^2+2+1$. But both words dominate $2^{k-1}+2^{k-3}+\ldots+2+1$ and this word is dominated by $2^{k-1}+2^{k-2}+2^{k-4}+\ldots+2^2+1$. This is a contradiction, so there is no PDS in $B^*(2,4l)$, and this proves the case *a*).

If $k=4l+1$ we need an "island" of size 4 in a D. We explain this case *b*) for k=5 only. Eg. $01010 \in D$. 01010 dominates the set $01010, 10100, 10101, 00101$. How can we dominate 11010 and 01011? These are dominated by 10101 but this word is already dominated. 11010 dominates the set $11010, 01101, 11101, 10101, 10100$, the last two are dominated by 01010. 01011 dominates the set $01011, 00101, 10101, 10110, 10111$, the second and third words are dominated by 01010. So we put eg. 11101 in D! Now we can not to put 10111 in D, because these two words dominate each other. Try to put 10110 in D! The problem is that 11011 is dominated by both 11101 and 10110. This contradiction proves the case $k=5$, and we can speak about the case $k=4l+1$ similarly in general.

In case *c*) if we have a D, eg. $2^{k-2}+2^{k-4}+\ldots+1 \in D$, because must be an "island" in size 4. $2^{k-2}+2^{k-4}+\ldots+1$ dominates the set $2^{k-2}+2^{k-4}+\ldots+1$, $2^{k-1}+2^{k-3}+\ldots+2$, $2^{k-3}+2^{k-5}+\ldots+2$, $2^{k-1}+2^{k-3}+\ldots+2+1$. We need to dominate $2^{k-2}+2^{k-4}+\ldots+2^2$ and $2^{k-1}+2^{k-2}+2^{k-4}+\ldots+2^2+1$, too (these are two still not dominated neighbors of $2^{k-1}+2^{k-3}+\ldots+2$). But all the potential words $2^{k-1}+2^{k-3}+\ldots+2^3+1$ or $2^{k-1}+2^{k-3}+\ldots+2^3$ to $2^{k-2}+2^{k-4}+\ldots+2^2$ and $2^{k-2}+2^{k-3}+2^{k-5}\ldots+2$ or $2^{k-1}+2^{k-2}+2^{k-4}+\ldots+2$ to $2^{k-1}+2^{k-2}+2^{k-4}+\ldots+2^2+1$ dominate the same word, it is $2^{k-1}+2^{k-2}+2^{k-4}\ldots+2^2$. This is a contradiction again.

The last case *d*) is very similar to the case *a*), so we do not deal with it.

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